Generalization of the Agranovich - Toshich transformation and a constraint free bosonic representation for systems of truncated oscillators

1996 J. Phys. A: Math. Gen. 29 L23

(http://iopscience.iop.org/0305-4470/29/2/001)

J. Phys. A: Math. Gen. **29** (1996) L23–L29. Printed in the UK

LETTER TO THE EDITOR

Generalization of the Agranovich–Toshich transformation and a constraint free bosonic representation for systems of truncated oscillators

A V Ilinskaia† and K N Ilinski‡§

† Institute of Physics, St-Petersburg University, Universitetskii prosp. 2, 198904 St-Petersburg, Russia

‡ School of Physics and Space Research, University of Birmingham, Birmingham B15 2TT, UK

Received 18 October 1995

Abstract. The generalization of the Agranovich–Toshich representation of Paulion operators in terms of bosonic ones for the case of truncated oscillators of higher ranks is derived. We use this generalization to introduce a new constraint-free bosonic description of truncated oscillator systems. The corresponding functional integral representations for thermodynamic quantities are given and the application to investigations of long-range order in the system is discussed.

About three decades ago Agranovich and Toshich [1] proposed a bosonic expression for the creation and annihilation operators of Paulions (i.e. particles obeying fermionic anticommutation relations on same the site and the bosonic commutation relations on different sites; alternatively, it is possible to realize them as lattice spins $\frac{1}{2}$ or truncated oscillators of rank 2). Explicitly, the equation has the form

$$\hat{P}_i^{\,+} = b_i^+ \sqrt{\sum_{k=0}^{\infty} \frac{(-2)^k}{(k+1)!} (b_i^+)^k b_i^k} \qquad \hat{P}_i = (\hat{P}_i^+)^+ \,. \tag{1}$$

Here $\hat{P}_i^+, \hat{P}_i$ are the creation and annihilation operators of a Paulion on site i, which obey the Paulionic commutation relations

$$\begin{aligned} &\hat{P}_i^+ \hat{P}_i + \hat{P}_i \hat{P}_i^+ = 1 \qquad (\hat{P}_i^+)^2 = \hat{P}_i^{\,2} = 0 \\ &[\hat{P}_i^+, \hat{P}_j] = [\hat{P}_i, \hat{P}_j] = 0 \qquad \text{for} \quad i \neq j \end{aligned} \tag{2}$$

and b_i^+, b_i are the bosonic operators on ith site.

Particles with anticommutation relations (2) on the same site and bosonic commutation relations on different sites occur widely in spin lattices, magnetic systems, models describing excitons in molecular crystals, defectons in quantum crystals and many others. In the original paper [1] Frenkel excitons were considered in connection with the possibility of their Bose condensation. As is usual in the problem of Bose condensation, the definition of an auxiliary bosonic description of the system is a central point because then the standard theory of a non-ideal Bose gas can be applied.

§ E-mail address: kni@th.ph.bham.ac.uk

0305-4470/96/020023+07$19.50

On the other hand, there are several applications of high-rank truncated oscillators. For example, it was recently shown that such operators can be used for the second quantization of particles with Haldane exclusion statistics [2]. Truncated oscillators also find applications in nonlinear optics, semiconductors, parasupersymmetric theories and other fields. That is why it seems to be interesting to generalize the Agranovich–Toshich description of truncated oscillators of rank 2 to higher ranks and investigate the corresponding bosonic representation.

As in the approach of the sigma model with a Wess–Zumino term [4] we will treat the constraint on the number of particles on each site exactly. To do this we will use the mapping of the orthogonal sum of identical copies of the truncated oscillator space of states to the bosonic space of states. In this mapping the creation and annihilation operators of truncated bosons are represented in the form of a power series of the usual bosonic creation and annihilation operators. This compels us to deal with infinite series of different vertices in the diagram technique. The choice of relevant contributions in such series should be dictated as usual by features of the concrete problem.

The letter is constructed as follows. We first prove the generalization of the Agranovich–Toshich equation for the case of an arbitrary rank of truncated oscillators. We give both variants of the mapping—with and without the square root (which corresponds to the equation proposed by Chernyak [3] for the Paulionic case; in practical use the latter is even more convenient). Then we describe an associated bosonic system for the case of many degrees of freedom and give the functional integral representation of thermodynamic quantities of the system. Several remarks conclude the letter.

Our first goal is to express the creation and annihilation operators of truncated bosons B^+, B of rank m with the algebra

$$BB^+ - B^+B = 1 - \frac{m}{(m-1)!}(B^+)^{m-1}B^{m-1} \qquad (B^+)^+ = B \qquad (B^+)^m = B^m = 0$$

in terms of the standard bosonic creation and annihilation operators b^+, b. In this section only one degree of freedom is considered but the generalization to many degrees of freedom is straightforward and will be considered in the next section. First of all we will construct the number particle operator of the truncated bosons $\hat{N}$ using the following operator:

$$1 + q^{(b^+b-k)} + q^{2(b^+b-k)} + \cdots + q^{(m-1)(b^+b-k)}$$

where $q = \exp\bigl(\mathrm{i}2\pi/m\bigr)$ and m is the rank of the truncated bosons. One can prove that this operator does not equal zero only on states $|lm+k\rangle$, where $l = 0, 1, 2, \ldots$. So the operator $\hat{N} = B^+B$ can be expressed as follows:

$$\begin{aligned}\hat{N} = 0\cdot\Bigl(1 + q^{b^+b} + q^{2b^+b} + \cdots + q^{(m-1)b^+b}\Bigr)\frac{1}{m} \\ +1\cdot\Bigl(1 + q^{(b^+b-1)} + q^{2(b^+b-1)} + \cdots + q^{(m-1)(b^+b-1)}\Bigr)\frac{1}{m} \\ + \cdots + (m-1)\Bigl(1 + q^{(b^+b-m+1)} + q^{2(b^+b-m+1)} + \cdots + q^{(m-1)(b^+b-m+1)}\Bigr)\frac{1}{m}\,.\end{aligned}$$

Summing terms of the same order one can obtain the following expression for the operator $\hat{N}$:

$$\hat{N} = \sum_{k=1}^{m-1}\frac{q^k}{1-q^k}q^{kb^+b} + \frac{m-1}{2}\,. \tag{3}$$

Now to order bosonic operators we can use the equation for the normal ordered exponent

$$q^{kb^+b} = \exp\left(\mathrm{i}\frac{2\pi}{m}kb^+b\right) =: \exp(q^k - 1)b^+b :$$

where $: \cdots :$ denotes normal ordering. Then expression (3) takes the form

$$\hat{N} = \sum_{l=1}^{\infty}\sum_{k=1}^{m-1}\frac{(-1)^l}{l!}q^k(1-q^k)^{l-1}(b^+)^l b^l \, . \tag{4}$$

Now let us assume that the creation (annihilation) operators of truncated bosons can be written in the following form:

$$B^+ = b^+\sqrt{\sum_{k=0}^{\infty}\alpha_k (b^+)^k b^k} \qquad B = (B^+)^+$$

and assume that the α_k are real. Then, using expression (4) for the number particle operator $\hat{N} = B^+B$, one can obtain the expression for the coefficients α_k as

$$\alpha_l = \sum_{k=1}^{m-1}\frac{(-1)^{l+1}}{(l+1)!}q^k(1-q^k)^l \, .$$

For $m = 2$ ($q = -1$) the coefficients take the form

$$\alpha_l = \frac{(-2)^l}{(l+1)!}$$

which gives us the Agranovich–Toshich representation (1) for the creation (annihilation) operators of truncated bosons of rank 2. So we have proved the generalization of Agranovich–Toshich equation for truncated oscillators of higher ranks:

$$B^+ = b^+\sqrt{\sum_{k=0}^{\infty}\sum_{k=1}^{m-1}\frac{(-1)^{l+1}}{(l+1)!}q^k(1-q^k)^l(b^+)^k b^k} \qquad B = (B^+)^+ \qquad q = \mathrm{e}^{\mathrm{i}2\pi/m} \, .$$

Let us now 'take the square root' in equation (5). To do this we will follow the method proposed by Chernyak in [3]. The main point of the method is to use the projection operator on the vacuum state of the auxiliary boson system, i.e. on the vector $|0\rangle$. To express this projection operator $\mathcal{P}$ in terms of b^+, b the coherent state representation is convenient

$$|z\rangle = \exp(-\tfrac{1}{2}\bar{z}z)\exp(zb^+)|0\rangle$$

$$\langle z'|z\rangle = \exp(-\tfrac{1}{2}\bar{z}'z' - \tfrac{1}{2}\bar{z}z + \bar{z}'z) \, .$$

From the last relations one easily derives

$$\langle z'|\mathcal{P}|z\rangle = \exp(-\bar{z}'z)\langle z'|z\rangle \, .$$

On the other hand, for any k, l we have

$$\langle z'|(b^+)^k b^l|z\rangle = (\bar{z}')^k z^l \langle z'|z\rangle \, .$$

Then we can conclude that

$$\langle z'|\mathcal{P}|z\rangle = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\langle z'|(b^+)^l b^l|z\rangle \, .$$

This means that the projection operator has the following expression in terms of the bosonic creation and annihilation operators:

$$\mathcal{P} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} (b^+)^l b^l \equiv : \exp(-b^+ b) : \, . \tag{5}$$

We now can use equation (5) to construct the creation and annihilation operators B^+, B which obey the algebra (3). Indeed, it is easy to check from the matrix form that the following relations hold:

$$\begin{aligned} B^+ &= \sum_{n=0}^{\infty} \sum_{k=0}^{m-2} (b^+)^{mn+k+1} \mathcal{P} b^{mn+k} \frac{\sqrt{k+1}}{(mn+k)!\sqrt{mn+k+1}} \\ B &= \sum_{n=0}^{\infty} \sum_{k=0}^{m-2} (b^+)^{mn+k} \mathcal{P} b^{mn+k+1} \frac{\sqrt{k+1}}{(mn+k)!\sqrt{mn+k+1}} \, . \end{aligned} \tag{6}$$

It is obvious that relations (6) satisfy the algebra (3). On the other hand, the operators given by relations (5) satisfy the same algebra and have the same matrix form. Hence we see that equations (6) corresond to the 'taking of the square root' in (5). For the particular case $m = 2$ our equations reduce to the equations originally obtained by Chernyak [3] for the case of Paulionic operators.

We will now describe the mapping from the system of truncated oscillators to the auxiliary bosonic system. The goal is to escape the introduction of a constraint. To do this we will embed an infinite number of copies of the finite dimensional space of states in the bosonic space of states and then proceed with the consideration of this new (auxiliary) bosonic space.

To explain this in detail, let us first of all consider one degree of freedom (i.e. a single site). Then the creation B^+ and annihilation B operators have the following matrix form in the m-dimensional Hilbert space of states $\mathcal{H}_B$ (m is a rank of the truncated oscillator):

$$B^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{3} & 0 & \cdots & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & \sqrt{m-1} & 0 \end{pmatrix}$$

$$B = (B^+)^{\mathrm{Tr}}$$

with basis $\{|0\rangle, |1\rangle, \ldots, |m-1\rangle\}$ and the obvious notations. Now we introduce the infinite orthogonal sum $\mathcal{H}_b = \oplus \sum_{n=0}^{\infty} \mathcal{H}_{B,n}$ of such m-dimensional Hilbert spaces $\mathcal{H}_{B,n}$ with basis $\{\{|0\rangle, |1\rangle, \ldots, |m-1\rangle\}, \ldots, \{|nm+1\rangle, |nm+2\rangle, \ldots, |nm+m-1\rangle\}, \ldots\}$. The extensions of the creation and annihilation operators B^+, B in this space have the form:

$$\hat{B}^+ = \mathrm{diag}(B^+, B^+, \ldots) \qquad \hat{B} = \mathrm{diag}(B, B, \ldots) \, .$$

It follows that all thermodynamic quantities calculated with operators $\hat{B^+}$, $\hat{B}$ are exactly the same as those calculated with the original operators B^+, B. Indeed, for example,

$$\langle \hat{B}^+ \hat{B} \rangle \equiv \frac{Sp(\hat{B}^+ \hat{B} e^{-\beta(E-\mu)\hat{B}^+\hat{B}})}{Sp(e^{-\beta(E-\mu)\hat{B}^+\hat{B}})}$$

is identical to the same expressions without hats due to the block structure of our operators (we should add that the partition functions differ by an infinite numerical constant which does not affect observable physical quantities). The conclusion is still valid if we start with a lattice of truncated oscillators and then introduce hats for the operators.

We have found above the corresponding expressions for the creation and annihilation operators $\hat{B}^+$, $\hat{B}$ in terms of the bosonic creation and annihilation operators b^+, b acting in the Hilbert space $\mathcal{H}_b$. These are given by equation (5) with the square root or by equation (6) without it (which we will use below).

The equations considered above in this letter can be applied to construct the Hamiltonian of the auxiliary bosonic system. So if we start with the following Hamiltonian H_t of truncated oscillators on a lattice:

$$H_t = \sum_i \Delta B_i^+ B_i + \sum_{i\neq j} M_{i,j} B_i^+ B_j + \sum_{i\neq j} (L_{i,j} B_i^+ B_j^+ + \text{HC}) + \sum_{i\neq j} J_{i,j} B_i^+ B_i B_j^+ B_j$$

then the corresponding Hamiltonian of the auxiliary bosons has the form:

$$H = \sum_i \Delta \sum_{l=0}^{\infty} a(l)(b_i^+)^{l+1} b_i^{l+1} + \sum_{i\neq j} M_{i,j} b_i^+ S_{ij} b_j$$

$$+ \sum_{i\neq j} (L_{i,j} b_i^+ b_j^+ S_{ij} + \text{HC}) + \sum_{i\neq j} J_{i,j} \sum_{l,m=0}^{\infty} a(l)a(m)(b_i^+)^{l+1}(b_j^+)^{m+1} b_i^{l+1} b_i^{m+1} .$$

Here the following notation has been introduced:

$$S_{i,j} = \sum_{l,m=0}^{\infty} A(l)A(m)(b_i^+)^l (b_j^+)^m b_i^l b_j^m$$

and the coefficients $A(l)$ and $a(l)$ are defined as

$$A(l) \equiv \sum_{k=0}^{min(m-2,l)} \sum_{n=0}^{\left[\frac{l-k}{m}\right]} \frac{(-1)^{l-mn-k}}{(l-mn-k)!} \frac{\sqrt{k+1}}{(mn+k)!\sqrt{mn+k+1}}$$

and

$$a(l) \equiv \frac{(-1)^{l+1}}{(l+1)!} \sum_{k=1}^{m-1} q^k (1-q^k)^l .$$

Let us note once more that the system with Hamiltonian H is equivalent to the original Hamiltonian of truncated bosons H_t and does not require any additional constraint.

Using the standard procedure, we can write down the functional integral representation of the partition function and correlators of the auxiliary bosonic system and the original system of truncated oscillators. For example, according to the definition and equation (6), the following relations arise:

$$Z \equiv Sp(e^{-\beta H}) = \int \mathrm{D}b^+(\tau)\, \mathrm{D}b(\tau)\, \mathrm{e}^S$$

$$\langle B_i^+ B_j \rangle = \int \mathrm{D}b^+(\tau)\, \mathrm{D}b(\tau) \sum_{l,m=0}^{\infty} A(l)A(m)(b_i^+(\tau))^{l+1}(b_j^+(\tau))^m b_i^l(\tau)) b_j^{m+1}(\tau) \mathrm{e}^S / Z$$

where the action S is defined by the form of the Hamiltonian H:

$$S = \int_0^\beta \Bigg(\sum_i \frac{\partial b_i^+(\tau)}{\partial \tau} b_i(\tau) - \sum_i \Delta \sum_{l=0}^{\infty} a(l)(b_i^+(\tau))^{l+1} b_i^{l+1}(\tau) + \sum_{i\neq j} M_{i,j} b_i^+(\tau) S_{ij}(\tau) b_j(\tau) + \sum_{i\neq j}(L_{i,j} b_i^+(\tau) b_j^+(\tau) S_{ij}(\tau) + \mathrm{HC}) + \sum_{i\neq j} J_{i,j} \sum_{l,m=0}^{\infty} a(l)a(m)(b_i^+(\tau))^{l+1}(b_j^+(\tau))^{m+1} b_i^{l+1}(\tau) b_j^{m+1}(\tau) \Bigg)\, \mathrm{d}\tau\ .$$

All other correlators can be obtained in the same manner and give us the bosonic functional integral representation which is free of constraints and limiting procedures (how it would be if we considered an analog of the hard-core interaction on site and took a limit). The functional integral form then allows the simplest approach to the derivation of diagram technique rules which are standard ones for the problems in question. It is tempting to note that this technique is much less complicated and much more straightforward than the spin operator technique and is very natural for the consideration of problems concerning Bose condensation (long-range order) in the system just using the standard Bogoliubov approach to the subject.

In summary, in this letter we have considered the generalization of the Agranovich–Toshich representation for the creation and annihilation operators of truncated oscillators in terms of auxiliary bosons. This allowed us to formulate a model of interacting bosons with the equivalent thermodynamic behaviour and express various correlators of a truncated oscillator system through series of correlators of interacting bosons. It is important to note that such a description is free of constraints or limiting procedures which occur in other approaches.

Moreover, this technique can be applied to the high-spin systems or Hubbard-like models using the obvious transformation of truncated oscillator operators to the corresponding spin operators or the Hubbard operators. This allows one to escape the complicated operator technique and make use of standard diagrammatic methods alone.

However, we have to note the difficulties which arise in this framework. The method leads to infinite series of types of interactions and thus infinite series of the various vertices in the diagrams. Which contribution is relevant has to be determined by the concrete physical problem where several assumptions about the structure of a ground state and excitations have to be made. Of course this problem occurs in any perturbation theory! We think that the technique described above could be convenient in the consideration of questions concerning the existence of Long Range Order in a system which is equivalent to Bose condensation. Indeed the interacting boson picture seems to be most natural for such investigation.

We want to thank V M Agranovich, J M F Gunn and M W Long for discussions about the problem. This work was supported by the Grant of Russian Fund of Fundamental Investigations N 95-01-00548, the Euler stipend of the German Mathematical Society, INTAS-939 and by UK EPSRC Grant GR/J35221. We are grateful for the hospitality to the International Center for Theoretical Physics where the work was finished.

References

[1] Agranovich V M and Toshich B S 1968 *Sov. Phys.–JETP* **26** 104

[2] Ilinskaya A V, Ilinski K N and Gunn G M F 1995 Fractional dimensional Hilbert space, second quantization and dynamical interaction for particles with Halsane's exclusion statistics *Preprint* TPBU-95-5 (*Phys. Rev.* B to appear)
[3] Chernyak V Ya 1992 *Phys. Lett.* **163A** 117
[4] Wiegmann P 1988 *Phys. Rev. Lett.* **60** 821
Fradkin E 1991 *Field Theories of Condensed Matter Systems* (Reading, MA: Addison-Wesley)